Homework 03

Section 12.1 – Vectors in the Plane

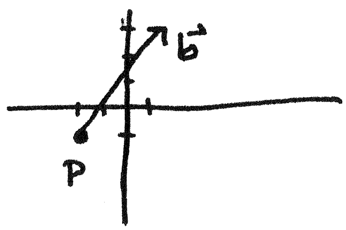
Exercises 2, 4, 7, 13, 17, 18, 25, 29, 37, 39, 41, 43, 46, 57

2. Sketch the vector $\mathbf{b} = \langle 3, 4 \rangle$ based at $P = (-2, -1)$.

end point has

x-coord $= -2 + 3 = 1$

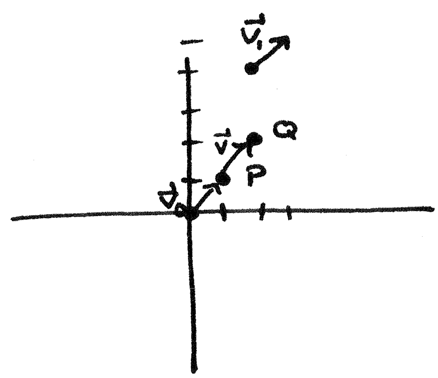
y-coord $= -1 + 4 = 3$



4. Let $\mathbf{v} = \overrightarrow{PQ}$, where $P = (1, 1)$ and $Q = (2, 2)$. What is the head of the vector $\mathbf{v}_1$ equivalent to $\mathbf{v}$ based at $(2, 4)$? What is the head of the vector $\mathbf{v}_0$ equivalent to $\mathbf{v}$ based at the origin? Sketch $\mathbf{v}$, $\mathbf{v}_0$, and $\mathbf{v}_1$.

$\vec{v} = \langle 2-1, 2-1 \rangle = \langle 1, 1 \rangle$

- start at $(2,4) \Rightarrow$ end at $(3,5)$
- Start at origin $\Rightarrow$ end at $(1,1)$



In Exercises 5–8, find the components of $\overrightarrow{PQ}$.

7. $P = (3, 5)$, $Q = (1, -4)$

$\overrightarrow{PQ} = \langle 1-3, -4-5 \rangle = \langle -2, -9 \rangle$

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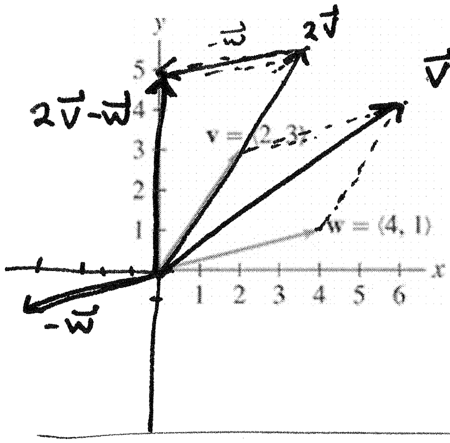
13. $\left\langle -\frac{1}{2}, \frac{5}{3} \right\rangle + \left\langle 3, \frac{10}{3} \right\rangle$ add component-by-component

$\left\langle -\frac{1}{2} + 3, \frac{5}{3} + \frac{10}{3} \right\rangle$

17. Sketch 2**v**, −**w**, **v** + **w**, and 2**v** −**w** for the vectors in Figure 23.



$2\vec{v} = 2\langle 2,3\rangle = \langle 4,6\rangle$

$-\vec{w} = -1\cdot\langle 4,1\rangle = \langle -4,-1\rangle$

$\vec{v}+\vec{w} = \langle 2,3\rangle + \langle 4,1\rangle = \langle 6,4\rangle$

$2\vec{v}-\vec{w} = \langle 4,6\rangle - \langle 4,1\rangle = \langle 0,5\rangle$

Figure 23

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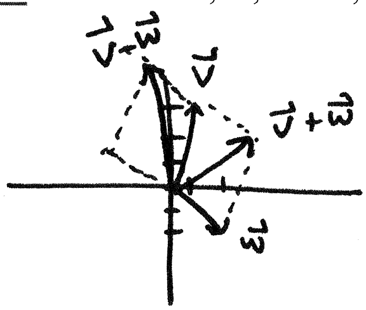
18. Sketch **v** = ⟨1, 3⟩, **w** = ⟨2, −2⟩, **v** + **w**, **v** −**w**.



$\vec{v}-\vec{w} = \langle 1,3\rangle - \langle 2,-2\rangle$

$= \langle 1,3\rangle + \langle -2,2\rangle$

$= \langle -1,5\rangle$

$\vec{v}+\vec{w} = \langle 3,1\rangle$

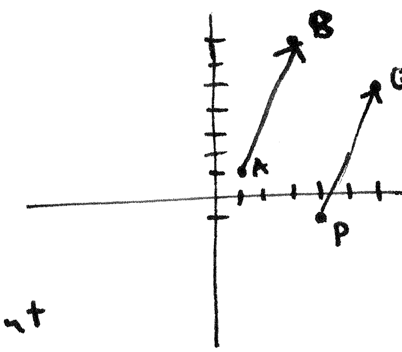
In Exercises 25–28, sketch the vectors $\overrightarrow{AB}$ and $\overrightarrow{PQ}$, and determine whether they are equivalent.

25. $A = (1, 1)$, $B = (3, 7)$, $P = (4, -1)$, $Q = (6, 5)$

$\overrightarrow{AB} = \langle 3-1, 7-1\rangle = \langle 2,6\rangle$

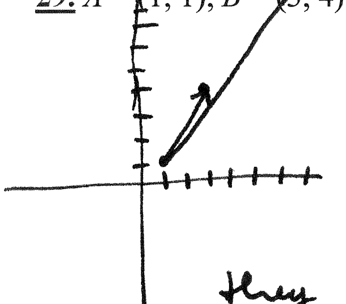
$\overrightarrow{PQ} = \langle 6-4, 5-(-1)\rangle = \langle 2,6\rangle$

same components ⇒ are equivalent



In Exercises 29–32, are $\overrightarrow{AB}$ and $\overrightarrow{PQ}$ parallel? And if so, do they point in the same direction?

29. $A = (1, 1)$, $B = (3, 4)$, $P = (1, 1)$, $Q = (7, 10)$



$\overrightarrow{AB} = \langle 4-1, 3-1\rangle = \langle 3,2\rangle$

$\overrightarrow{PQ} = \langle 7-1, 10-1\rangle = \langle 6,9\rangle$

Not multiples of eachother ⇒ not parallel

they are NOT parallel

In Exercises 37–42, find the given vector.

37. Unit vector $\mathbf{e_v}$ where $\mathbf{v} = \langle 3, 4 \rangle$

$$\|\vec{v}\| = \sqrt{3^2+4^2} = \sqrt{9+16} = \sqrt{25} = 5$$

$$e_{\vec{v}} = \frac{1}{\|\vec{v}\|} \cdot \vec{v} = \frac{1}{5} \cdot \langle 3,4 \rangle$$

$$e_{\vec{v}} = \left\langle \frac{3}{5}, \frac{4}{5} \right\rangle$$

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39. Vector of length 4 in the direction of $\mathbf{u} = \langle -1, -1 \rangle$

$$\|\vec{u}\| = \sqrt{(-1)^2+(-1)^2} = \sqrt{1+1} = \sqrt{2}$$

unit vector in direction $\vec{u}$ $= e_{\vec{u}} = \frac{1}{\sqrt{2}} \cdot \langle -1, -1 \rangle = \left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle$

vector of length 4 in direction $\vec{u}$
$$\begin{cases} 4 \cdot e_{\vec{u}} = 4 \cdot \left\langle \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}} \right\rangle \\ \quad = \left\langle \frac{-4}{\sqrt{2}}, \frac{-4}{\sqrt{2}} \right\rangle \end{cases}$$

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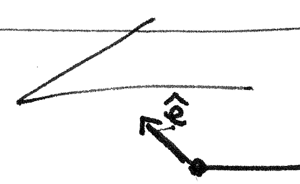
41. Unit vector $\mathbf{e}$ making an angle of $\frac{4\pi}{7}$ with the x-axis



ONE such vector is

one vector making this $\measuredangle$ is

$$\vec{u} = \left\langle \cos\left(\frac{4\pi}{7}\right), \sin\left(\frac{4\pi}{7}\right) \right\rangle$$

check $\|\vec{u}\| = \sqrt{\cos^2(\pi/4) + \sin^2(\pi/4)} = \sqrt{1} = 1$

NOTICE $\vec{u}$ is already a unit vector, as desired.

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43. Find all scalars λ such that $\lambda \langle 2, 3 \rangle$ has length 1.

$$\|\langle 2,3 \rangle\| = \sqrt{2^2+3^2} = \sqrt{4+9} = \sqrt{13}$$

to get vector of length 1, need $\lambda = \frac{1}{\sqrt{13}}$ or $\lambda = \frac{-1}{\sqrt{13}}$

$\uparrow$ unit vector in same direction

$\uparrow$ unit vector in other direction

46. What are the coordinates *a* and *b* in the parallelogram in Figure 25(B)?

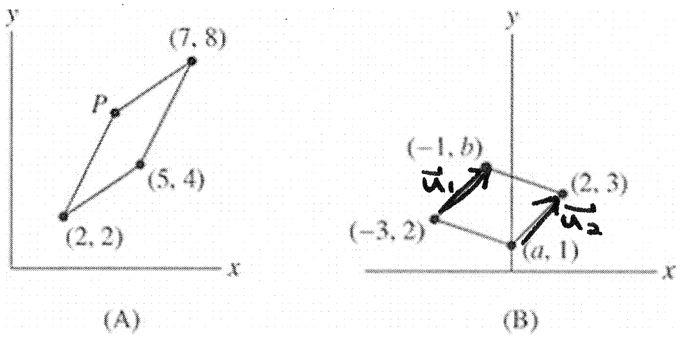


Figure 25

know $\vec{u_1} = \vec{u_2}$

compute $\vec{u_1} = \langle -1-(-3), b-2 \rangle = \langle 2, b-2 \rangle$

$\vec{u_2} = \langle 2-a, 3-1 \rangle = \langle 2-a, 2 \rangle$

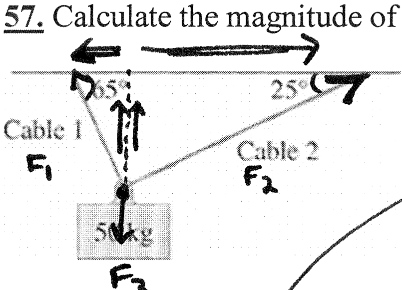
conclude $\vec{u_1}$ & $\vec{u_2}$ equivalent

$\Rightarrow$ have same components

$\Rightarrow \quad 2 = 2-a \quad \& \quad b-2 = 2$

$a = 0 \qquad b = 4$

57. Calculate the magnitude of the force on cables 1 and 2 in Figure 27. (fun application!)



up/down forces cancel

$50 = F_3 = F_1 \cdot \sin(65°) + F_2 \cdot \cos(25°)$

System of 2 eqns in 2 unknowns

left/right forces cancel

$0 = F_1 \cdot \cos(65°) - F_2 \cdot \cos(25°)$

$F_1 \cdot \cos(65°) = F_2 \cdot \cos(25°)$

$F_1 = F_2 \cdot \frac{\cos(25°)}{\cos(65°)}$

conclude

$50 = F_1 \cdot \sin(65°) + F_2 \cdot \cos(25°)$ &

$50 = F_2 \cdot \frac{\cos(25°)}{\cos(65°)} \cdot \sin(65) + F_2 \cdot \cos(25)$

$50 = F_2 \left(\frac{\cos(25)}{\cos(65)} \cdot \sin(65) + \cos(25) \right)$

$F_2 = \frac{50}{\frac{\cos(25)}{\cos(65)} \cdot \sin(65) + \cos(25)}$

& can plug in to find F_1 also!

Section 12.2 -- Vectors in Three Dimensions

6, 9, 12, 24, 26, 31, 35, 37, 42, 43, 50, 62

In Exercises 5–8, find the components of the vector $\overrightarrow{PQ}$.

6. $P = (-3, -4, 2)$, $Q = (1, -4, 3)$

$$\overrightarrow{PQ} = \langle 1-(-3), -4-(-4), 3-2 \rangle$$
$$= \langle 4, 0, 1 \rangle$$

In Exercises 9–12, let $R = (1, 4, 3)$.

9. Calculate the length of $\overrightarrow{OR}$.

$$\|\overrightarrow{OR}\| = \|\langle 1, 4, 3 \rangle\|$$
$$= \sqrt{1^2 + 4^2 + 3^2}$$
$$= \sqrt{1 + 16 + 9}$$
$$= \sqrt{26}.$$

12. Find the components of $\mathbf{u} = \overrightarrow{PR}$, where $P = (1, 2, 2)$.

$$\vec{u} = \overrightarrow{PR} = \langle 1-1, 4-2, 3-2 \rangle$$
$$\vec{u} = \langle 0, 2, 1 \rangle$$

In Exercises 24–27, find the given vector.

24. $\mathbf{e_v}$, where $\mathbf{v} = \langle 1, 1, 2 \rangle$

$$\|\vec{v}\| = \sqrt{1^2+1^2+2^2} = \sqrt{1+1+4} = \sqrt{6}$$

$$e_{\vec{v}} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{6}}\langle 1,1,2\rangle$$

$$e_{\vec{v}} = \left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$$

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26. Unit vector in the direction of $\mathbf{u} = \langle 1, 0, 7 \rangle$

$$\|\vec{u}\| = \sqrt{1^2+0^2+7^2}$$

$$= \sqrt{1+49}$$

$$= \sqrt{50}$$

$$e_{\vec{u}} = \frac{1}{\|\vec{u}\|}\vec{u}$$

$$= \frac{1}{\sqrt{50}}\langle 1,0,7\rangle$$

$$e_{\vec{u}} = \left\langle \frac{1}{\sqrt{50}}, 0, \frac{7}{\sqrt{50}} \right\rangle$$

In Exercises 29–36, find a vector parametrization for the line with the given description.

31. Passes through $P = (4, 0, 8)$, direction vector $\mathbf{v} = 7\mathbf{i} + 4\mathbf{k}$

$\vec{v} = \langle 7, 4, 0\rangle$, $\vec{p} = \overrightarrow{OP} = \langle 4, 0, 8\rangle$

$F(t) = \langle 4, 0, 8\rangle + t \cdot \langle 7, 4, 0\rangle$

35. Passes through O and $(4, 1, 1)$

$\vec{v} = \overrightarrow{O(4,1,1)} = \langle 4, 1, 1\rangle$ $\qquad \vec{p} = \overrightarrow{OO} = \langle 0, 0, 0\rangle$

$F(t) = \langle 0, 0, 0\rangle + t\langle 4, 1, 1\rangle$

$= t \cdot \langle 4, 1, 1\rangle$

In Exercises 37–40, find parametric equations for the lines with the given description.

37. Perpendicular to the xy-plane, passes through the origin

x-y plane is $\perp$ to $\langle 0, 0, 1\rangle = \hat{k}$

$\Rightarrow \vec{v} = \langle 0, 0, 1\rangle$

$\vec{p} = \langle 0, 0, 0\rangle$

$\Rightarrow F(t) = t\langle 0, 0, 1\rangle = \langle 0, 0, t\rangle$

42. Find a parametrization of the line through $P = (4, 9, 8)$ perpendicular to the yz-plane.

yz plane is $\perp$ to $\langle 1, 0, 0\rangle = \hat{i}$

$\Rightarrow \vec{v} = \langle 1, 0, 0\rangle$

$\vec{p} = \overrightarrow{OP} = \langle 4, 9, 8\rangle$

$\Rightarrow F(t) = \langle 4, 9, 8\rangle + t\langle 1, 0, 0\rangle$

Aside:
can also be written
$F(t) = \langle 4+t, 9, 8\rangle$

Section 12.3 – The Dot Product

In Exercises 1–12, compute the dot product.

1. $\langle 1, 2, 1\rangle \cdot \langle 4, 3, 5\rangle$

$= 1\cdot 4 + 2\cdot 3 + 5\cdot 1$

$= 4 + 6 + 5$

$= 15$

2. $\langle 3, -2, 2\rangle \cdot \langle 1, 0, 1\rangle$

$= 3\cdot 1 + (-2)\cdot 0 + 2\cdot 1$

$= 3 + 0 + 2$

$= 5$

3. $\langle 0, 1, 0\rangle \cdot \langle 7, 41, -3\rangle$

$= 0\cdot 7 + 1\cdot 41 + 0\cdot(-3)$

$= 41$

4. $\langle 1, 1, 1\rangle \cdot \langle 6, 4, 2\rangle$

$= 1\cdot 6 + 1\cdot 4 + 1\cdot 2$

$= 6 + 4 + 2$

$= 12$

5. $\langle 3, 1\rangle \cdot \langle 4, -7\rangle$

$= 3\cdot 4 + 1\cdot(-7)$

$= 12 - 7$

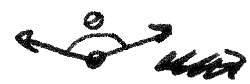
$= 5$

In Exercises 13–18, determine whether the two vectors are orthogonal and, if not, whether the angle between them is acute or obtuse.

<u>13.</u> $\langle 1, 1, 1\rangle, \langle 1, -2, -2\rangle$

$\langle 1,1,1\rangle \cdot \langle 1,-2,-2\rangle = 1\cdot 1 + 1\cdot(-2) + 1\cdot(-2) = 1-2-2 = -3$

dot product neg $\Rightarrow$ $\cos\theta$ is neg

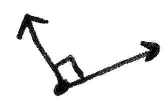


$\Rightarrow$ angle is obtuse

i.e. $\pi/2 < \theta \leq \pi$

<u>14.</u> $\langle 0, 2, 4\rangle, \langle -5, 0, 0\rangle$

$\langle 0,2,4\rangle \cdot \langle -5,0,0\rangle = 0\cdot(-5) + 2\cdot 0 + 4\cdot 0 = 0$



dot product 0 $\Rightarrow$ cos is zero

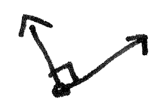
$\Rightarrow$ $\theta = \pi/2$

$\Rightarrow$ vectors are orthogonal $(\perp)$

<u>15.</u> $\langle 1, 2, 1\rangle, \langle 7, -3, -1\rangle$

$\langle 1,2,1\rangle \cdot \langle 7,-3,-1\rangle = 7 - 6 - 1 = 0$

dot product 0 $\Rightarrow$ $\theta = \pi/2$

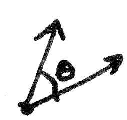


$\Rightarrow$ vectors are orthogonal.

<u>16.</u> $\langle 0, 2, 4\rangle, \langle 3, 1, 0\rangle$

$\langle 0,2,4\rangle \cdot \langle 3,1,0\rangle = 0 + 2 + 0 = 2$

dot product positive $\Rightarrow$ $\cos\theta$ is positive



$\Rightarrow$ angle is acute

i.e. $0 \leq \theta < \pi/2$

In Exercises 19–22, find the cosine of the angle between the vectors.

19. $\langle 0, 3, 1\rangle, \langle 4, 0, 0\rangle$

Remember $\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$

so $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\cdot\|\vec{v}\|}$

$\langle 0,3,1\rangle \cdot \langle 4,0,0\rangle$

$= 0+0+0$

$= 0$

so $\cos\theta = \dfrac{0}{\|\vec{u}\|\cdot\|\vec{v}\|} = 0$

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20. $\langle 1, 1, 1\rangle, \langle 2, -1, 2\rangle$

$\vec{u}\cdot\vec{v} = \|\vec{u}\|\cdot\|\vec{v}\|\cdot\cos\theta$

$\vec{u}\cdot\vec{v} = \langle 1,1,1\rangle\cdot\langle 2,-1,2\rangle = 2-1+2 = 3$

$\|\vec{u}\| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

$\|\vec{v}\| = \sqrt{2^2+(-1)^2+2^2} = \sqrt{4+1+4}$

$= \sqrt{9} = 3$

$3 = \sqrt{3}\cdot 3\cdot\cos\theta$

$\Rightarrow \cos\theta = \dfrac{1}{\sqrt{3}}$

In Exercises 51–58, find the projection of **u** *along* **v**.

$\mathrm{proj}_{\vec{v}}\vec{u} = \underbrace{\|\vec{u}\|\cdot\cos\theta}_{\text{magnitude of component}}\cdot\underbrace{\dfrac{1}{\|\vec{v}\|}\cdot\vec{v}}_{e_{\vec{v}}}$

or derive

$= \cancel{\|\vec{u}\|}\cdot\dfrac{\vec{u}\cdot\vec{v}}{\cancel{\|\vec{u}\|}\cdot\|\vec{v}\|}\cdot\dfrac{1}{\|\vec{v}\|}\cdot\vec{v}$

Remember

↳ $\mathrm{Proj}_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{(\|\vec{v}\|)^2}\cdot\vec{v}$

51. $\mathbf{u} = \langle 2, 5\rangle$, $\mathbf{v} = \langle 1, 1\rangle$

$\vec{u}\cdot\vec{v} = 2\cdot1+5\cdot1 = 7$

$\|v\| = \sqrt{1^2+1^2} = \sqrt{2}$

$\mathrm{Proj}_{\vec{v}}\vec{u} = \dfrac{7}{(\sqrt{2})^2}\cdot\langle 1,1\rangle = \left\langle \dfrac{7}{2}, \dfrac{7}{2}\right\rangle$

52. $\mathbf{u} = \langle 2, -3\rangle$, $\mathbf{v} = \langle 1, 2\rangle$

$\vec{u}\cdot\vec{v} = 2\cdot1+-3\cdot2 = 2-6 = -4$

$\|\vec{v}\| = \sqrt{1^2+2^2} = \sqrt{5}$

$\mathrm{Proj}_{\vec{v}}\vec{u} = \dfrac{-4}{(\sqrt{5})^2}\cdot\langle 1,2\rangle = \left\langle \dfrac{-4}{5}, \dfrac{-8}{5}\right\rangle$

Remember: $\text{Proj}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|^2}\cdot\vec{v}$

53. $\mathbf{u} = \langle -1, 2, 0\rangle$, $\mathbf{v} = \langle 2, 0, 1\rangle$

$\vec{u}\cdot\vec{v} = (-1)\cdot 2 + 2\cdot 0 + 0\cdot 1 = -2$

$\|\vec{v}\| = \sqrt{2^2+0^2+1^2} = \sqrt{5}$

$\text{Proj}_{\vec{v}}\vec{u} = \frac{-2}{(\sqrt{5})^2}\cdot\langle 2,0,1\rangle = \langle \frac{-4}{5}, 0, \frac{-2}{5}\rangle$

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54. $\mathbf{u} = \langle 1, 1, 1\rangle$, $\mathbf{v} = \langle 1, 1, 0\rangle$

$\vec{u}\cdot\vec{v} = 1\cdot 1 + 1\cdot 1 + 1\cdot 0 = 2$

$\|\vec{v}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$\text{Proj}_{\vec{v}}\vec{u} = \frac{2}{(\sqrt{2})^2}\langle 1,1,0\rangle$

$= \langle 1,1,0\rangle$

In Exercises 59 and 60, compute the component of **u** *along* **v**.

$\text{comp}_{\vec{v}}\vec{u} = \|u\|\cdot\cos\theta$

and $\vec{u}\cdot\vec{v} = \|u\|\cdot\|v\|\cdot\cos\theta$

$\Rightarrow \|u\|\cdot\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$

59. $\mathbf{u} = \langle 3, 2, 1\rangle$, $\mathbf{v} = \langle 1, 0, 1\rangle$

$\vec{u}\cdot\vec{v} = 3\cdot 1 + 2\cdot 0 + 1\cdot 1 = 4$

$\|\vec{v}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

Remember: $\text{Comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$

$\text{comp}_{\vec{v}}\vec{u} = \frac{4}{\sqrt{2}}$

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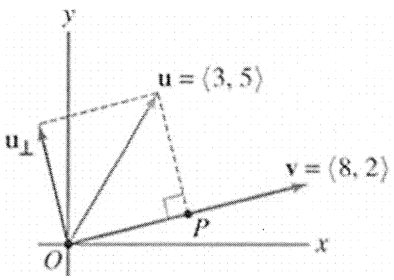
60. $\mathbf{u} = \langle 3, 0, 9\rangle$, $\mathbf{v} = \langle 1, 2, 2\rangle$

$\vec{u}\cdot\vec{v} = 3\cdot 1 + 0\cdot 2 + 9\cdot 2 = 3+18 = 21$

$\|v\| = \sqrt{1^2+2^2+2^2} = \sqrt{9} = 3$

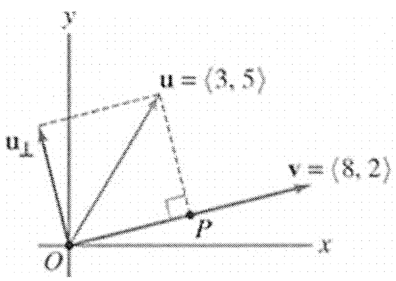
$\text{Comp}_{\vec{v}}\vec{u} = \frac{21}{3} = 7$

61. Find the length of $\overline{OP}$ in Figure 14.



$$\text{comp}_{\vec{v}}\vec{u} = \frac{\vec{u}\cdot\vec{v}}{\|\vec{v}\|}$$

61. Find the length of $\overline{OP}$ in Figure 14.



length of $\overline{OP}$ = component of $\vec{u}$ in direction $\vec{v}$

$\vec{u}\cdot\vec{v} = \langle 3,5\rangle\cdot\langle 8,2\rangle = 3\cdot 8 + 5\cdot 2$

$= 64 + 10 = 74$

$\|\vec{v}\| = \sqrt{8^2+2^2} = \sqrt{68}$

length of $\overline{OP}$ is $\frac{74}{\sqrt{68}}$

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7. $\mathbf{k}\cdot\mathbf{j} = 0$

8. $\mathbf{k}\cdot\mathbf{k} = 1$

9. $(\mathbf{i}+\mathbf{j})\cdot(\mathbf{j}+\mathbf{k})$ $\overset{\text{FOIL}}{=}$ $\hat{i}\hat{j} + \hat{i}k + \hat{j}\hat{j} + \hat{j}\hat{k}$

$= 0 + 0 + 1 + 0$

$= 1$

10. $(3\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}-4\mathbf{k})$ $\overset{\text{FOIL}}{=}$ $3\hat{j}\hat{i} - 12\hat{k}\hat{j} + 2\hat{i}\hat{k} - 8\hat{k}\hat{k}$

$= 0 - 0 + 0 - 8$